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A COMPACT DIFFERENCE–FINITE VOLUME SCHEME FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

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We propose a method that uses fourth order accurate staggered mesh compact differences for the momentum equations, and a fourth order accurate integral type finite volume scheme for the continuity equation, describe a new set of intermediate boundary conditions of the Runge–Kutta method, its accuracy is one order higher than that of the conventional method. For a 2D traveling wave calculation, its numerical results are much better than those of the conventional method. Calculation results for a scalar equation and a 2D traveling wave flow and a square driven cavity(Re=100) show fourth order accuracy of the staggered mesh compact scheme. Numerical results in the square driven cavity are obtained for Re=1000,7500 as well as Re=100. We get steady solutions for Re=7500. Our numerical results support the conclusion that the Hopf bifurcation point R_c is not lower than 7500.

Key Words: compact difference scheme, finite volume scheme, incompressible N–S equations, intermediate boundary conditions, Runge–Kutta method

1. INTRODUCTION

We consider the compact difference schemes that use not only values of the function itself, but also those of its derivatives as unknowns.[16][17][14][6] They are of high order accuracy with less grid points, a better stability, a better resolution for high frequency waves, and fewer boundary difference points than traditional methods. One application is the direct numerical simulation (DNS) of model turbulent flows, which is very difficult to simulate, requires all the relevant scales to be properly represented in the numerical method. [13][14] applied compact schemes to the DNS, obtained good results.

Liu[11] constructed a compact scheme according to [5], applied it to simulating the driven flow problem, in the calculation he found oscillations near the left–upper corner. When Re much larger, there were oscillations near a lower corner too, so that the computations could not keep on. This shows that the central difference regular grid compact scheme produces non–physical numerical oscillations at where flow parameter varies acutely. He[11] applied an upwind technique to the non–staggered mesh compact difference scheme to solve the incompressible flow, solved the above oscillations problem successfully.

We use the staggered mesh compact scheme[12] which does not produce the above oscillations in the same calculations of the driven flow problem without using the upwind technique. The three–point central difference scheme is of the fourth order accuracy, while the upwind three–point compact scheme is of third order accuracy. These show good qualities of the staggered mesh method.

We propose a method that uses the fourth order accurate staggered mesh compact differences (which we presented in [8],[12]) for the momentum equations(§3), and uses an integral type finite volume scheme for the continuity equation(§2). The driven flow problem in a square cavity with Re=7500 is calculated. We get steady solutions, while some other authors got unsteady results[1][11].

The unsteady viscous Navier–Stokes equations (momentum equations) with primitive variables velocity and pressure are:

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{A}(\boldsymbol{V}) + \boldsymbol{\nabla} p = \boldsymbol{0}, \text{ in } \Omega \ , \quad (\text{ where } \boldsymbol{A}(\boldsymbol{V}) = (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} - \nu \nabla^2 \boldsymbol{V} \ ) \tag{1.1}$$

The continuity equation for incompressible fluid flows is:

$$\text{div} \boldsymbol{V} = 0, \quad \text{ in } \Omega \qquad (\boldsymbol{V} = (u, v)^T \text{ for a 2D case }) \tag{1.2}$$

We consider an explicit discrete form of (1.1)(1.2):

$$\frac{\boldsymbol{V}^{n+1} - \boldsymbol{V}^*}{\Delta t} + \boldsymbol{\nabla}_h p^{n+1} = 0, \quad (\text{ where } \boldsymbol{V}^* = \boldsymbol{V}^n - \Delta t \boldsymbol{A}_h(\boldsymbol{V}^n) \ ), \tag{1.3}$$

$$\text{div}_h \boldsymbol{V}^{n+1} = 0, \qquad (\ \boldsymbol{V}^{n+1} = (u^{n+1}, v^{n+1})^T \text{ for a 2D case }), \tag{1.4}$$

here it is the first order accurate for the time discretization.

Since unsteady phenomenon is concerned, it is necessary to approximate the time derivative with high order accuracy. We use the fourth order accurate Runge–Kutta method for unsteady problems, see §4.

2. AN INTEGRAL TYPE FINITE VOLUME SCHEME FOR THE CONTINUITY EQUATION

In "§5 of [8]" we have pointed out that such approximation, (the fourth order accurate staggered mesh compact difference for momentum equations, and the integral type scheme "(5.2)(5.1) of [8]" for the continuity equation), is another fourth order accurate difference scheme as well as the compact schemes (for all equations) presented in [8].

Now we describe the integral type finite volume scheme (i. e.,(2.3)–(2.5) below) for the continuity equation (1.2) on $\Omega = (0, L^x) \times (0, L^y)$. Consider a uniform grid. $\Delta x = L^x/N, \Delta y = L^y/M$. Integrating (1.2) on $e = ((i-1)\Delta x, i\Delta x) \times ((j-1)\Delta y, j\Delta y)$ results

$$\int_{\partial e} \boldsymbol{V} \cdot \boldsymbol{n} \mathrm{d}\gamma = 0, \tag{2.1}$$

where ∂e is the boundary of e, $\boldsymbol{V} = (u, v)^T$, $\boldsymbol{n}$ is the outward normal vector on e. Set

$$\bar{u}_{i,j-\frac{1}{2}} = \frac{1}{\Delta y}\int_{(j-1)\Delta y}^{j\Delta y} u(i\Delta x, y)\mathrm{d}y, \quad \bar{v}_{i-\frac{1}{2},j} = \frac{1}{\Delta x}\int_{(i-1)\Delta x}^{i\Delta x} v(x, j\Delta y)\mathrm{d}x, \tag{2.2}$$

then (2.1) can be written by

$$\frac{\bar{u}_{i,j-\frac{1}{2}} - \bar{u}_{i-1,j-\frac{1}{2}}}{\Delta x} + \frac{\bar{v}_{i-\frac{1}{2},j} - \bar{v}_{i-\frac{1}{2},j-1}}{\Delta y} = 0, \qquad \begin{matrix} 1 \le i \le N \\ 1 \le j \le M \end{matrix} \tag{2.3}$$

The following approximations are fourth order accurate

$$\bar{u}_{i,j-\frac{1}{2}} = u_{i,j-\frac{1}{2}} + \frac{1}{24}(u_{i,j-\frac{3}{2}} - 2u_{i,j-\frac{1}{2}} + u_{i,j+\frac{1}{2}}), \qquad \begin{matrix} 1 \le i \le N-1 \\ 2 \le j \le M-1 \end{matrix} \tag*{(2.4)$_1$}$$

$$\bar{v}_{i-\frac{1}{2},j} = v_{i-\frac{1}{2},j} + \frac{1}{24}(v_{i-\frac{3}{2},j} - 2v_{i-\frac{1}{2},j} + v_{i+\frac{1}{2},j}), \qquad \begin{matrix} 2 \le i \le N-1 \\ 1 \le j \le M-1 \end{matrix} \tag*{(2.4)$_2$}$$

where $u_{i,j-\frac{1}{2}} = u(i\Delta x, (j-\frac{1}{2})\Delta y), v_{i-\frac{1}{2},j} = v((i-\frac{1}{2})\Delta x, j\Delta y)$, then we get **an integral type scheme (2.3)(2.4)** if we use the same notations $\bar{u}_{i,j-\frac{1}{2}}$, $\bar{v}_{i-\frac{1}{2},j}$, $u_{i,j-\frac{1}{2}}$, $v_{i-\frac{1}{2},j}$ for the discretized variables. Near the boundary, we use the third order accurate approximations:

$$\bar{u}_{i,\frac{1}{2}} = u_{i,\frac{1}{2}} + \frac{1}{18}(2u^{\Gamma}_{i,0} - 3u_{i,\frac{1}{2}} + u_{i,\frac{3}{2}}) \qquad (1 \le i \le N) \tag*{(2.5)$_1$}$$

$$\bar{u}_{i,M-\frac{1}{2}} = u_{i,M-\frac{1}{2}} + \frac{1}{18}(2u^{\Gamma}_{i,M} - 3u_{i,M-\frac{1}{2}} + u_{i,M-\frac{3}{2}}) \quad (1 \le i \le N) \tag*{(2.5)$_2$}$$

$$\bar{v}_{\frac{1}{2},j} = v_{\frac{1}{2},j} + \frac{1}{18}(2v^{\Gamma}_{0,j} - 3v_{\frac{1}{2},j} + v_{\frac{3}{2},j}) \quad (1 \le j \le M) \tag*{(2.5)$_3$}$$

$$\bar{v}_{N-\frac{1}{2},j} = v_{N-\frac{1}{2},j} + \frac{1}{18}(2v^{\Gamma}_{N,j} - 3v_{N-\frac{1}{2},j} + v_{N-\frac{3}{2},j}) \quad (1 \le j \le M) \tag*{(2.5)$_4$}$$

$u^{\Gamma}_{i,0} = u^{\Gamma}(i\Delta x, 0), u^{\Gamma}_{i,M} = u^{\Gamma}(i\Delta x, L^y), v^{\Gamma}_{0,j} = v^{\Gamma}(0, j\Delta y), v^{\Gamma}_{N,j} = v^{\Gamma}(L^x, j\Delta y)$.
On the boundary, $\bar{u}, \bar{v}$ are averages of integration, e. g.,

$$\bar{u}_{0,j-\frac{1}{2}} = \frac{1}{\Delta y}\int_{(j-1)\Delta y}^{j\Delta y} u(0, y)\mathrm{d}y \tag{2.6}$$

Note that truncation errors of (2.3) for $i = 1, N$ are one order lower accuracy than those for $1 < i < N$.

The integral type scheme (2.3)–(2.5) for the continuity equation contains the following **advantages** over the compact schemes (presented in "§3 of [12]" and "§2 of [8]"):
1. It has more physics meaning: $\bar{u}_{i-1,j-\frac{1}{2}}\Delta y\Delta t$ is the quantity that flows in through the left edge of the element e; ...; $\bar{v}_{i-\frac{1}{2},j}\Delta x\Delta t$ is flow–out quantity through the upper egde.

Then (2.3) means the sum of the flow–in quantity is equal to the sum of the flow–out quantity in the element $e = ((i-1)\Delta x, i\Delta x) \times ((j-1)\Delta y, j\Delta y)$;
2. The stream function can be obtained directly by

$$\psi_{i,0} = 0, \quad \psi_{i,j} = \psi_{i,j-1} + \bar{u}_{i,j-\frac{1}{2}}\Delta y, \quad i = 1, 2, ..., N-1; j = 1, 2, ..., M-1; \tag{2.7}$$

$\bar{u}_{i,j-\frac{1}{2}}$ can be obtained from (2.4) (2.5);
3. Boundary velocity affects the flow directly through the continuity equation;

4. At the boundary point where the velocity discontinues, the scheme decreases the error of the compact scheme (3.8)(3.15) for the continuity equation. ((3.8)(3.15) are "(2.8)(2.15) of [8]", "(3.8)(3.15) of [12]"). For the driven flow problem in §6.3, u discontinues at points (0,1) and (1,1).

3. STAGGERED MESH COMPACT DIFFERENCE SCHEMES

We consider the two dimensional unsteady viscous incompressible N–S equations (1.1)(1.2) with

$$\boldsymbol{A}(\boldsymbol{V}) = (uu_x + vu_y - \nu(u_{xx} + u_{yy}), uv_x + vv_y - \nu(v_{xx} + v_{yy}))^T, \tag{3.1}$$

$$\boldsymbol{\nabla} p = (p_x, p_y)^T, \mathrm{div}\boldsymbol{V} = u_x + v_y \tag{3.2}$$

3.1 Fourth Order Accurate Compact Schemes on a 2D Staggered Mesh

For this kind of schemes, the derivatives, as well as the velocity and the pressure themselves, are employed to be unknowns of the difference equations. For explicit schemes and those that have no implicit compact difference, we can solve the derivatives u_x, u_y from u first. The solution pattern is similar to the ADI method: in x and y directions, solve the partial derivatives in x and y directions respectively. Consider a finite difference scheme of (3.1)(3.2),

$$\frac{\boldsymbol{V}^{n+1} - \boldsymbol{V}^n}{\Delta t} + \boldsymbol{A}_h(\boldsymbol{V}^n) + \boldsymbol{\nabla}_h p^{n+1} = 0, \tag{3.3}$$

$$\mathrm{div}_h \boldsymbol{V}^{n+1} = 0, \tag{3.4}$$

here $\boldsymbol{V}^n = (u^n, v^n)^T$; $\boldsymbol{A}_h, \boldsymbol{\nabla}_h$, div_h are finite difference forms of $\boldsymbol{A}, \boldsymbol{\nabla}$, div on a uniform grid:
(1) finite difference forms of the first derivatives u_x, u_y in $\boldsymbol{A}_h(\boldsymbol{V})$ are:

$$\frac{u'_{i-1,j-\frac{1}{2}} + 4u'_{i,j-\frac{1}{2}} + u'_{i+1,j-\frac{1}{2}}}{6} = \frac{u_{i+1,j-\frac{1}{2}} - u_{i-1,j-\frac{1}{2}}}{2\Delta x}, \begin{array}{l} 1 \le i \le N-1 \\ 1 \le j \le M \end{array}, \tag{3.5$_1$}$$

$$\frac{u'_{i,j-\frac{3}{2}} + 4u'_{i,j-\frac{1}{2}} + u'_{i,j+\frac{1}{2}}}{6} = \frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{3}{2}}}{2\Delta y}, \begin{array}{l} 1 \le i \le N-1 \\ 2 \le j \le M-1 \end{array}, \tag{3.5$_2$}$$

v_y, v_x in $\boldsymbol{A}_h(\boldsymbol{V})$ are similar to (3.5),
(2) finite difference forms of the second derivatives u_{xx}, u_{yy} in $\boldsymbol{A}_h(\boldsymbol{V})$:

$$\frac{u''_{i-1,j-\frac{1}{2}} + 10u''_{i,j-\frac{1}{2}} + u''_{i+1,j-\frac{1}{2}}}{12} = \frac{u_{i-1,j-\frac{1}{2}} - 2u_{i,j-\frac{1}{2}} + u_{i+1,j-\frac{1}{2}}}{(\Delta x)^2}, \tag{3.6$_1$}$$

$$(1 \le i \le N-1, \quad 1 \le j \le M)$$

$$\frac{u''_{i,j-\frac{3}{2}} + 10u''_{i,j-\frac{1}{2}} + u''_{i,j+\frac{1}{2}}}{12} = \frac{u_{i,j-\frac{3}{2}} - 2u_{i,j-\frac{1}{2}} + u_{i,j+\frac{1}{2}}}{(\Delta y)^2}, \begin{array}{l} 1 \le i \le N-1 \\ 2 \le j \le M-1 \end{array} \tag{3.6$_2$}$$

v_{yy}, v_{xx} in $\boldsymbol{A}_h(\boldsymbol{V})$ are similar to (3.6),

(3) for the derivative p_x in $\boldsymbol{\nabla}_h p$,

$$\frac{p'_{i-1,j-\frac{1}{2}} + 22p'_{i,j-\frac{1}{2}} + p'_{i+1,j-\frac{1}{2}}}{24} = \frac{p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}, \quad \begin{array}{l} 2 \le i \le N-2 \\ 1 \le j \le M \end{array} \tag{3.7}$$

similar for p_y,

(4) compact difference for u_x in $\mathrm{div}_h \boldsymbol{V}$,(in this paper we mainly use another scheme for $\mathrm{div}_h \boldsymbol{V}$ described in §2).

$$\frac{u'_{i-\frac{3}{2},j-\frac{1}{2}} + 22u'_{i-\frac{1}{2},j-\frac{1}{2}} + u'_{i+\frac{1}{2},j-\frac{1}{2}}}{24} = \frac{u_{i,j-\frac{1}{2}} - u_{i-1,j-\frac{1}{2}}}{\Delta x}, \quad \begin{array}{l} 2 \le i \le N-1 \\ 1 \le j \le M \end{array} \tag{3.8}$$

similar for v_y in $\mathrm{div}_h \boldsymbol{V}$ in v_y,

(5) for u in uv_x in $(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}$, the interpolation is employed:

$$u_{i-\frac{1}{2},j} = \frac{\tilde{u}_{i-1,j-\frac{1}{2}} + \tilde{u}_{i,j-\frac{1}{2}} + \tilde{u}_{i-1,j+\frac{1}{2}} + \tilde{u}_{i,j+\frac{1}{2}}}{4}, \quad \begin{array}{l} 1 \le i \le N \\ 1 \le j \le M-1 \end{array}, \tag{3.9$_1$}$$

$$\tilde{u}_{i,j-\frac{1}{2}} = u_{i,j-\frac{1}{2}} - \frac{(\Delta x)^2 (u_{xx})_{i,j-\frac{1}{2}} + (\Delta y)^2 (u_{yy})_{i,j-\frac{1}{2}}}{8}, \quad \begin{array}{l} 0 \le i \le N \\ 1 \le j \le M \end{array} \tag{3.9$_2$}$$

where u_{xx}, u_{yy} can adopt the results of (3.6)(3.12)(3.13). v in vu_y is similar.

(6) for $\boldsymbol{V}_t$: (3.3) adopts the first order accurate difference. A fourth order accurate Runge–Kutta method is given in §4.

Remark 3.1 If we change $u, u', i, \Delta x$ to

$$\int_{(j-1)\Delta y}^{y} u(i\Delta x, y)\mathrm{d}y, \quad u,\ j,\ \Delta y,$$

then (3.8) becomes (2.4)$_1$ (with (2.2)), (3.15) becomes (2.5)$_1$ with LHS and RHS reversed.

3.2 Difference Formulations Near the Boundary

$\boldsymbol{V}|_\Gamma = \boldsymbol{V}_\Gamma = (u^\Gamma, v^\Gamma)^T$ on $\Gamma = \partial\Omega$. Now we mainly describe difference formulations near the left boundary $x = 0$.

(1)$_1$ $(u_x)_{0,j-\frac{1}{2}} = (-v_y^\Gamma)_{0,j-\frac{1}{2}}, (u_x)_{N,j-\frac{1}{2}} = (-v_y^\Gamma)_{N,j-\frac{1}{2}}, \ (1 \le j \le M)$ (3.10)

(1)$_2$ u_y at $y = \frac{3}{4}\Delta y$ and $y = L^y - \frac{3}{4}\Delta y$,

$$\frac{3u'_{i,\frac{1}{2}} + u'_{i,\frac{3}{2}}}{4} = \frac{u_{i,\frac{3}{2}} - u_{i,0}}{\frac{3}{2}\Delta y}, \frac{u'_{i,M-\frac{3}{2}} + 3u'_{i,M-\frac{1}{2}}}{4} = \frac{u_{i,M} - u_{i,M-\frac{3}{2}}}{\frac{3}{2}\Delta y}, \tag{3.11}$$

$$(1 \le i \le N-1)$$

here $u_{i,0} = u^\Gamma(i\Delta x, 0), u_{i,M} = u^\Gamma(i\Delta x, L^y)$;

v_x at $x = \frac{3}{4}\Delta x$ and $x = L^x - \frac{3}{4}\Delta x$ similar,

(2)$_1$ u_{xx} at $x = \frac{2}{3}\Delta x$:

$$\frac{u''_{0,j-\frac{1}{2}} + 2u''_{1,j-\frac{1}{2}}}{3} = \frac{1}{\Delta x}\left(\frac{u_{2,j-\frac{1}{2}} - u_{0,j-\frac{1}{2}}}{2\Delta x} - u'_{0,j-\frac{1}{2}}\right), \ (1 \le j \le M) \tag{3.12}$$

where $u'_{0,j-\frac{1}{2}} = -(v_y^\Gamma)_{0,j-\frac{1}{2}} = -v_y^\Gamma(0,(j-\frac{1}{2})\Delta y)$;

u_{xx} at $x = L^x - \frac{10}{9}\Delta x$, v_{yy} at $y = \frac{10}{9}\Delta y$ and v_{yy} at $y = L^y - \frac{10}{9}\Delta y$ are similar,

$(2)_2$ u_{yy} at $y = \frac{2}{3}\Delta y$:

$$\frac{5u''_{i,\frac{1}{2}} + u''_{i,\frac{3}{2}}}{6} - \frac{1}{48}(u''_{i,\frac{1}{2}} - 2u''_{i,\frac{3}{2}} + u''_{i,\frac{5}{2}}) = \frac{4}{3}\cdot\frac{2u_{i,0} - 3u_{i,\frac{1}{2}} + u_{i,\frac{3}{2}}}{(\Delta y)^2}, \quad (1 \le i \le N-1) \tag{3.13}$$

where $u_{i,0} = u^\Gamma_{i,0}$;

u_{yy} at $y = L^y - \frac{2}{3}\Delta y$, v_{xx} at $x = \frac{2}{3}\Delta x$ and v_{xx} at $x = L^x - \frac{2}{3}\Delta x$ are similar,

(3) p_x at $x = \Delta x$:

$$p'_{1,j-\frac{1}{2}} + \frac{p'_{1,j-\frac{1}{2}} - 2p'_{2,j-\frac{1}{2}} + p'_{3,j-\frac{1}{2}}}{24} = \frac{p_{\frac{3}{2},j-\frac{1}{2}} - p_{\frac{1}{2},j-\frac{1}{2}}}{\Delta x}, \ (1 \le j \le M) \tag{3.14}$$

p_x at $x = L^x - \Delta x$, p_y at $y = \Delta y$ and p_y at $y = L^y - \Delta y$ similar,

(4) u_x in div$\boldsymbol{V}$ at $x = \frac{1}{2}\Delta x$,(this is for (3.8)):

$$\frac{2u'_{0,j-\frac{1}{2}} + 15u'_{\frac{1}{2},j-\frac{1}{2}} + u'_{\frac{3}{2},j-\frac{1}{2}}}{18} = \frac{u_{1,j-\frac{1}{2}} - u_{0,j-\frac{1}{2}}}{\Delta x}, (1 \le j \le M), \tag{3.15}$$

where $u'_{0,j-\frac{1}{2}} = -(v_y^\Gamma)_{0,j-\frac{1}{2}}$; u_x in div$\boldsymbol{V}$ at $x = L^x - \frac{1}{2}\Delta x$,

v_y in div$\boldsymbol{V}$ at $y = \frac{1}{2}\Delta y$ and v_y in div$\boldsymbol{V}$ at $y = L^y - \frac{1}{2}\Delta y$ similar,

(5) boundary values of u_{xx}, u_{yy} in $(3.9)_2$: for $j = 1, 2, ..., M$,

$$(u_{xx})_{0,j-\frac{1}{2}} = 2(u_{xx})_{\frac{1}{2},j-\frac{1}{2}} - (u_{xx})_{1,j-\frac{1}{2}}, \tag*{$(3.16)_1$}$$

$$(u_{xx})_{\frac{1}{2},j-\frac{1}{2}} = ((u_x)_{1,j-\frac{1}{2}} - (u_x)_{0,j-\frac{1}{2}})/\Delta x, \tag*{$(3.16)_2$}$$

$$(u_{yy})_{0,j-\frac{1}{2}} = (u^\Gamma_{yy})_{0,j-\frac{1}{2}}, \tag*{$(3.16)_3$}$$

where $(u_x)_{1,j-\frac{1}{2}}$ can be got from $(3.5)_1$ (3.10), $(u_x)_{0,j-\frac{1}{2}} = -(v_y^\Gamma)_{0,j-\frac{1}{2}}$.

3.3 An Upwind Compact Difference Scheme

Alter the formulation $(3.5)_1$ to

$$\frac{1}{3}u'_{i-1,j-\frac{1}{2}} + \frac{2}{3}u'_{i,j-\frac{1}{2}} = \frac{u_{i+1,j-\frac{1}{2}} + 4u_{i,j-\frac{1}{2}} - 5u_{i-1,j-\frac{1}{2}}}{6\Delta x}, (\text{if } u_{i,j-\frac{1}{2}} \ge 0) \tag*{$(3.17)_1$}$$

$$\frac{2}{3}u'_{i,j-\frac{1}{2}} + \frac{1}{3}u'_{i+1,j-\frac{1}{2}} = \frac{5u_{i+1,j-\frac{1}{2}} - 4u_{i,j-\frac{1}{2}} - u_{i-1,j-\frac{1}{2}}}{6\Delta x}, (\text{if } u_{i,j-\frac{1}{2}} < 0) \tag*{$(3.17)_2$}$$

for $1 \le i \le N-1, 1 \le j \le M$. Alter $(3.5)_2$ similarly. Other formulations are same with (3.6)–(3.16), except $(3.16)_2$ altered to

$$(u_{xx})_{\frac{1}{2},j-\frac{1}{2}} = ((u_{2,j-\frac{1}{2}} + 4u_{1,j-\frac{1}{2}} - 5u_{0,j-\frac{1}{2}})/\Delta x - 6(u_x)_{0,j-\frac{1}{2}})/(4\Delta x) \tag{3.18}$$

The difference between (3.17) and "(2.8)(2.9) of [11]" is $F(= u')$ has no superscripts '+' and '–'.

3.4 Truncation Errors of the Discretizations

From the Taylor expansion, the truncation error (LHS minus RHS) is

$$\frac{1}{180}u_{x^5}(\Delta x)^4 \quad \text{for} \quad (3.5)_1; \qquad \frac{1}{240}u_{x^6}(\Delta x)^4 \quad \text{for} \quad (3.6)_1;$$

$$\frac{17}{5760}p_{x^5}(\Delta x)^4 \quad \text{for} \quad (3.7); \qquad \frac{17}{5760}u_{x^5}(\Delta x)^4 \quad \text{for} \quad (3.8);$$

$$\frac{1}{384}(5u_{x^4}(\Delta x)^4 + 6u_{xxyy}(\Delta x)^2(\Delta y)^2 + 5u_{y^4}(\Delta y)^4) \quad \text{for} \quad (3.9)_1 \qquad (\text{with} \quad (3.9)_2)$$

$$\frac{1}{64}u_{y^4}(\Delta y)^3 \quad \text{for} \quad (3.11); \qquad -\frac{1}{45}u_{x^5}(\Delta x)^3 \quad \text{for} \quad (3.12);$$

$$-\frac{1}{288}u_{y^5}(\Delta y)^3 \quad \text{for} \quad (3.13); \qquad \frac{1}{24}p_{x^4}(\Delta x)^3 \quad \text{for} \quad (3.14);$$

$$\frac{1}{144}u_{x^4}(\Delta x)^3 \quad \text{for} \quad (3.15); \qquad -\frac{1}{24}u_{x^4}(\Delta x)^2 \quad \text{for} \quad (3.16)_2; \qquad -\frac{1}{36}u_{x^4}(\Delta x)^3 \quad \text{for} \quad (3.17)_1;$$

$$-\frac{1}{12}u_{x^4}(\Delta x)^2 \text{ for } (3.18); \qquad -\frac{17}{5760}u_{y^4}(\Delta y)^4 \text{ for } (2.4)_1; \qquad -\frac{1}{144}u_{y^3}(\Delta y)^3 \quad \text{for} \quad (2.5)_1.$$

4. THE RUNGE–KUTTA METHOD FOR TIME DISCRETIZATIONS OF UNSTEADY PROBLEMS

Define $\boldsymbol{f}(\boldsymbol{V}) = \boldsymbol{A}_h(\boldsymbol{V}) + \boldsymbol{\nabla}_h p$, where $p = p(\boldsymbol{V})$ satisfies $\mathrm{div}_h(\boldsymbol{A}_h(\boldsymbol{V}) + \boldsymbol{\nabla}_h p) = 0$. Thus $\boldsymbol{f}$ is a function of $\boldsymbol{V}$. The fourth order Runge–Kutta formulation for solving $\boldsymbol{V}_t + \boldsymbol{f}(\boldsymbol{V}) = \boldsymbol{0}$ is:

$$\frac{\boldsymbol{V}^{n+1} - \boldsymbol{V}^n}{\Delta t} + \frac{\boldsymbol{f}(\boldsymbol{V}^n) + 2\boldsymbol{f}(\boldsymbol{V}^{n+\frac{1}{2}}) + 2\boldsymbol{f}(\bar{\boldsymbol{V}}^{n+\frac{1}{2}}) + \boldsymbol{f}(\bar{\boldsymbol{V}}^{n+1})}{6} = \boldsymbol{0}, \qquad (4.1)$$

where

$$\boldsymbol{V}^{n+\frac{1}{2}} = \boldsymbol{V}^n - \frac{\Delta t}{2}\boldsymbol{f}(\boldsymbol{V}^n), \; \bar{\boldsymbol{V}}^{n+\frac{1}{2}} = \boldsymbol{V}^n - \frac{\Delta t}{2}\boldsymbol{f}(\boldsymbol{V}^{n+\frac{1}{2}}), \; \bar{\boldsymbol{V}}^{n+1} = \boldsymbol{V}^n - \Delta t \boldsymbol{f}(\bar{\boldsymbol{V}}^{n+\frac{1}{2}}),$$

The Runge–Kutta method is widely used for time discretizations of high order accurate numerical methods of time–dependent problems. But the conventional method of intermediate boundary conditions has only first order accuracy[10] for time–dependent problems. The conventional method is

$$\boldsymbol{V}^{n+\frac{1}{2}} = \boldsymbol{V}|_{t=(n+\frac{1}{2})\Delta t}, \bar{\boldsymbol{V}}^{n+\frac{1}{2}} = \boldsymbol{V}|_{t=(n+\frac{1}{2})\Delta t}, \bar{\boldsymbol{V}}^{n+1} = \boldsymbol{V}|_{t=(n+1)\Delta t} \quad (\text{on } \Gamma) \qquad (4.2)$$

We have proposed the following formulations in [12] and in the journal 'Mathematica Numerica Sinica' (Vol. 20, No. 1, 1998, page 56):

$$\boldsymbol{V}^{n+\frac{1}{2}} = \boldsymbol{V}^n + \frac{\Delta t}{2}(\frac{\partial \boldsymbol{V}}{\partial t}\Big|_{t=n\Delta t}), \quad (\text{on } \Gamma) \qquad (4.3)_1$$

$$\bar{\boldsymbol{V}}^{n+\frac{1}{2}} = 2(\boldsymbol{V}|_{t=(n+\frac{1}{2})\Delta t}) - \boldsymbol{V}^{n+\frac{1}{2}}, \quad (\text{on } \Gamma) \qquad (4.3)_2$$

$$\bar{\boldsymbol{V}}^{n+1} = \boldsymbol{V}|_{t=(n+1)\Delta t}, \quad (\text{on } \Gamma) \tag{4.3$_3$}$$

Remark 4.1 (4.3) approximates "(22)–(24) of [10]" at least third order accurately. "(22)–(24) of [10]" are

$$v_0^1 = g(t) + \frac{\delta t}{2} g'(t) \tag{4.4$_1$}$$

$$v_0^2 = g(t) + \frac{\delta t}{2} g'(t) + \frac{(\delta t)^2}{4} g''(t) \tag{4.4$_2$}$$

$$v_0^3 = g(t) + \delta t g'(t) + \frac{(\delta t)^2}{2} g''(t) + \frac{(\delta t)^3}{4} g'''(t) \tag{4.4$_3$}$$

see [10] for details. In fact, $(4.4)_1$ is identical with $(4.3)_1$. If we use same notations as (4.4), then RHS of $(4.3)_2$ is $2g(t + \frac{1}{2}\delta t) - v_0^1 = v_0^2 + O((\delta t)^3)$, RHS of $(4.3)_3$ is $g(t + \delta t) = v_0^3 + O((\delta t)^3)$. Therefore, (4.4) and (4.3) are third order accurately approximate.

Remark 4.2 We use the iterative pressure Poisson equation method (§5) in the computational examples (§6) with $\boldsymbol{V}^* = \boldsymbol{V}^n - \frac{\Delta t}{2} \boldsymbol{A}_h(\boldsymbol{V}^n)$ for $\boldsymbol{V}^{n+\frac{1}{2}}$; $\boldsymbol{V}^* = \boldsymbol{V}^n - \frac{\Delta t}{2} \boldsymbol{A}_h(\boldsymbol{V}^{n+\frac{1}{2}})$ for $\bar{\boldsymbol{V}}^{n+\frac{1}{2}}$; $\boldsymbol{V}^* = \boldsymbol{V}^n - \Delta t \boldsymbol{A}_h(\bar{\boldsymbol{V}}^{n+\frac{1}{2}})$ for $\bar{\boldsymbol{V}}^{n+1}$

Remark 4.3 To solve the pressure, we can also use the pressure Poisson equation method (5.7)(3.1) with $\boldsymbol{V}^*$ same as those above.

5. THE ITERATIVE PRESSURE POISSON EQUATION METHOD

To solve the pressure, we use the iterative pressure Poisson equation method presented in [12], it uses the increment of the pressure, the difference of two successive pressures (p_k^{n+1} and p_{k+1}^{n+1}), instead of the pressure p^{n+1} as an unknown variable, see (5.3).

It has the following advantages:

1. it can ensure the discrete continuity equation satisfied as exactly as expected(see (5.4));

2. ∇_H^2 in the Poisson equation (5.3) can adopt a lower order accurate operator, e. g., for a 2D 4th order accurate compact scheme, ∇_H^2 can use the 5–point central difference.

3. it can be applied to three dimensional problems directly. (for a 3D problem, ∇_H^2 can adopt a 7–point central difference);

For the explicit discrete form (1.3)(1.4), the **pressure Poisson equation** can be written as:

$$\nabla_h^2 p^{n+1} = \frac{1}{\Delta t}\mathrm{div}_h \boldsymbol{V}^* \tag{5.1}$$

where $\boldsymbol{V}^* = \boldsymbol{V}^n - \Delta t \boldsymbol{A}_h(\boldsymbol{V}^n), \Delta t$ is the time step. Both LHS and RHS of (5.1) have various forms, corresponding to a kind of different pressure Poisson equation methods, cf. [3] [8]. [8] proposed a pressure Poisson equation that satisfies the equivalency (see (5.5)–(5.7)).

The iterative pressure Poisson equation method:

Take $n-$step values as initial: $\boldsymbol{V}_0^{n+1} = \boldsymbol{V}^n$, $p_0^{n+1} = p^n$, calculate $\boldsymbol{V}_{k+1}^{n+1}$, p_{k+1}^{n+1}, $k = 0, 1, 2, ...$ iteratively:

(1) calculate the velocity $\boldsymbol{V}_{k+1}^{n+1}$:

$$\frac{\boldsymbol{V}_{k+1}^{n+1} - \boldsymbol{V}^*}{\Delta t} + \boldsymbol{\nabla}_h p_k^{n+1} = 0, \quad (\text{ where } \boldsymbol{V}^* = \boldsymbol{V}^n - \Delta t \boldsymbol{A}_h(\boldsymbol{V}^n) \text{ }), \tag{5.2}$$

(2) solve the pressure p_{k+1}^{n+1} with an approximate Poisson equation (first solve $p_{k+1}^{n+1} - p_k^{n+1}$ as one unknown variable):

$$\nabla_H^2 (p_{k+1}^{n+1} - p_k^{n+1}) = \frac{1}{\Delta t}\mathrm{div}_h \boldsymbol{V}_{k+1}^{n+1} \tag{5.3}$$

(3) set $\boldsymbol{V}^{n+1} = \boldsymbol{V}_{k+1}^{n+1}$ when the following inequality valid

$$||\mathrm{div}_h \boldsymbol{V}_{k+1}^{n+1}|| \leq \epsilon \tag{5.4}$$

where $\epsilon > 0$ is a small quantity given beforehand, $||\cdot||$ is a norm. ϵ can be $O(h^4)$ when (2.3)(2.4) are of fourth order accuracy.

div_h in (5.3) (5.4) is a part of the original scheme, while $\boldsymbol{\nabla}_H^2$ in (5.3) can adopt a simple difference operator. e. g., for a 2D compact scheme, ∇_H^2 may adopt a 5–point central difference.

The iterative algorithm above is a means of solving the original scheme, such as the compact scheme. It does not change the numerical solution. (In this point, it just likes the Gauss–Seidel method for a system of linear equations).

Remark 5.1 The pressure Poisson equation (presented in [8]) that satisfies the equivalency is

$$D_h(\nabla_h p^{n+1}, \mathbf{0}, \mathbf{0}) = \frac{1}{\Delta t} D_h(\boldsymbol{V}^*, \boldsymbol{V}_\Gamma^{n+1}, \nabla \boldsymbol{V}_\Gamma^{n+1}) \tag{5.5}$$

this equation is derived from

$$D_h(\boldsymbol{V}^{n+1} - \alpha \mathbf{NS}_h, \boldsymbol{V}_\Gamma^{n+1}, \nabla \boldsymbol{V}_\Gamma^{n+1}) = 0 \tag{5.6}$$

where $\mathbf{NS}_h$ is the left hand side (LHS) of the discrete momentum equations (1.3), and

$$D_h(\boldsymbol{V}^{n+1}, \boldsymbol{V}_\Gamma^{n+1}, \nabla \boldsymbol{V}_\Gamma^{n+1}) = 0 \tag{5.7}$$

is rewritten from the discrete continuity equation (1.4) with the boundary conditions $\boldsymbol{V}^{n+1}|_\Gamma = \boldsymbol{V}_\Gamma^{n+1}, \quad \nabla \boldsymbol{V}^{n+1}|_\Gamma = \nabla \boldsymbol{V}_\Gamma^{n+1}$. Here $\boldsymbol{V}_\Gamma^{n+1} = \boldsymbol{V}_\Gamma|_{t=(n+1)\Delta t}$, $\nabla \boldsymbol{V}_\Gamma^{n+1} = \nabla \boldsymbol{V}_\Gamma|_{t=(n+1)\Delta t}$.

It is obvious that (5.6)(1.3) are equivalent to (5.7)(1.3). The equivalent equations (5.6)(1.3) can be applied to the four step fourth order accurate Runge–Kutta method with $\boldsymbol{V}^*$ valued as in Remark 4.2.

The compact schemes in §3 and §2 do not use all four components of $\nabla \boldsymbol{V}|_\Gamma$. They can only use the components which can get from $\boldsymbol{V}|_\Gamma = \boldsymbol{V}_\Gamma$ and the continuity equation (1.2). Those components are $\nabla(\boldsymbol{V}_\Gamma \cdot \boldsymbol{n})$ and $\nabla(\boldsymbol{V}_\Gamma \cdot \boldsymbol{\tau}) \cdot \boldsymbol{\tau}$, here $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are the unit normal and tangent vectors on Γ. In fact, (3.8) uses only $\nabla(\boldsymbol{V}_\Gamma \cdot \boldsymbol{n}) \cdot \boldsymbol{n}$. (2.3)–(2.5) (and other schemes generally) do not use $\nabla \boldsymbol{V}|_\Gamma$, i. e., no derivative boundary conditions in the discrete continuity equation.

RHS of (5.5) is $\frac{1}{\Delta t}\mathrm{div}_h \boldsymbol{V}^*$ (see (1.3) for $\boldsymbol{V}^*$) with $\boldsymbol{V}^*|_\Gamma = \boldsymbol{V}_\Gamma^{n+1}, \nabla \boldsymbol{V}^*|_\Gamma = \nabla \boldsymbol{V}_\Gamma^{n+1}$. LHS of (5.5) is $\mathrm{div}_h(\nabla_h p^{n+1})$ with $\nabla_h p^{n+1}|_\Gamma = \mathbf{0}$, $\nabla(\nabla_h p^{n+1})|_\Gamma = \mathbf{0}$. These numerical boundary conditions do not affect the result of p^{n+1} ("§6.3.1 of [9]" explained details of this for the MAC scheme).

6. NUMERICAL COMPUTATIONS

6.1 A Linear Scalar Case (with the compact scheme in §3)

We consider the scalar hyperbolic equation ((29)–(31) in [10])

$$u_t + u_x = 0, \quad u(0,t) = g = \sin(2\pi(-t)), u(x,0) = \sin(2\pi x) \tag{6.1}$$

$$\mathrm{Err}(N) = \frac{1}{N}\sqrt{\sum_{i=1}^{N} (u_i^n - u(i\Delta x, n\Delta t)^2)} \tag{6.2}$$

N: grid number, $\Delta x = 1/N, \Delta t = \mathrm{CFL}\Delta x$,
Conv. Rate: convergence rate (or convergence order):

$$\log_2(\mathrm{Err}(N/2)/\mathrm{Err}(N)) \tag{6.3}$$

Numerical results in the following table are obtained with the compact scheme $(3.5)_1$, Runge–Kutta method (4.1). (0.66498E–7 means 0.66498×10^{-7})

TABLE 1 (CFL=$\Delta t/\Delta x = 1, t = 1$)

	Compact A		Compact B		Compact C	
N	Err(N)	Conv Rate	Err(N)	Conv Rate	Err(N)	Conv Rate
256	0.66498E-7		0.53563E-7		0.60102E-7	
512	0.42889E-8	3.9546	0.33820E-8	3.9853	0.38073E-8	3.9806
1024	0.27997E-9	3.9373	0.21243E-9	3.9928	0.24083E-9	3.9827
2048	0.1872E-10	3.9028	0.1331E-10	3.9965	0.1529E-10	3.9774
4096	0.1298E-11	3.8507	0.8328E-12	3.9983	0.9800E-12	3.9637

Compact A: with the conventional boundary condition (4.2);
Compact B: with the boundary condition (4.4), (i. e., "(22)–(24) of [10])";
Compact C: with the boundary condition (4.3), (proposed by us).

Conclusions of this linear single equation calculations:

1. all these calculations obtain high order accurate solutions;
2. the convention method ("Compact A") is not bad for this case;
3. "Compact B" is a little better than "Compact C".

For a two dimensional nonlinear case, the conclusions are not very same. See next §6.2.

6.2 Two Dimensional Traveling Wave Calculations

A simple flow is considered for which the exact solution is known. The following is a 2D traveling wave solution of Navier–Stokes equations:

$$\begin{aligned} u &= 1 + 2\cos(2\pi(x-t))\sin(2\pi(y-t))e^{-8\pi^2\nu t} \\ v &= 1 - 2\sin(2\pi(x-t))\cos(2\pi(y-t))e^{-8\pi^2\nu t} \\ p &= (-\cos(4\pi(x-t)) - \cos(4\pi(y-t)))e^{-16\pi^2\nu t} \end{aligned} \tag{6.4}$$

$\boldsymbol{g} = (u,v)^T|_\Gamma, \Gamma = \partial\Omega, \Omega = (0,1)\times(0,1), (L^x = L^y = 1)$,

$$u(x,y,0) = 1 + 2\cos(2\pi x)\sin(2\pi y)$$

$$v(x,y,0) = 1 - 2\sin(2\pi x)\cos(2\pi y)$$

$$\mathrm{Err}(N) = \frac{1}{N}\sqrt{\sum_{i=1,j=1}^{N,N}((u^n_{i,j-\frac{1}{2}} - u^{n,\mathrm{accurate}}_{i,j-\frac{1}{2}})^2 + (v^n_{i-\frac{1}{2},j} - v^{n,\mathrm{accurate}}_{i-\frac{1}{2},j})^2)} \tag{6.5}$$

where

$$u^{n,\mathrm{accurate}}_{i,j-\frac{1}{2}} = u(i\Delta x, (j-\frac{1}{2})\Delta y, n\Delta t), \quad v^{n,\mathrm{accurate}}_{i-\frac{1}{2},j} = v((i-\frac{1}{2})\Delta x, j\Delta y, n\Delta t).$$

N: grid number, $\Delta x = \Delta y = 1/N, \Delta t = \mathrm{CFL}\Delta x$,
Conv. Rate: convergence rate (or convergence order), see (6.3).

Numerical results in the following tables 2 and 3 are obtained with the compact scheme (3.5)–(3.16), Runge–Kutta method (4.1), Re=100 and 500 respectively.

TABLE 2 (t=0.7, Re=100)

		Compact A		Compact B		Compact C	
N	dt/dx	Err(N)	Conv Rate	Err(N)	Conv Rate	Err(N)	Conv Rate
32	0.4	0.414E-4		0.204E-4		0.202E-4	
64	0.2	0.125E-5		0.755E-6		0.751E-6	
32	0.16	0.139E-4		0.139E-4		0.139E-4	
64	0.16	0.820E-6	4.0813	0.740E-6	4.2347	0.738E-6	4.2368
128	0.16	0.316E-6	1.3750	0.416E-7	4.1518	0.415E-7	4.1536
32	0.14	0.138E-4		0.139E-4		0.138E-4	
64	0.14	0.757E-6	4.1904	0.735E-6	4.2361	0.734E-6	4.2374
128	0.14	0.173E-6	2.1337	0.413E-7	4.1530	0.412E-7	4.1545

Compact A: with the conventional boundary condition (4.2);
Compact B: with the boundary condition (4.4), (i. e., "(22)–(24) of [10]");
Compact C: with the boundary condition (4.3), (proposed by us).

TABLE 3 (t=0.7, Re=500)

		Compact A		Compact B		Compact C	
N	dt/dx	Err(N)	Conv Rate	Err(N)	Conv Rate	Err(N)	Conv Rate
32	0.2	0.292E-4		0.291E-4		0.291E-4	
64	0.2	0.160E-5	4.1878	0.158E-5	4.1991	0.158E-5	4.1987
128	0.2	0.111E-6	3.8528	0.863E-7	4.1975	0.861E-7	4.1992
256	0.2	0.354E-7	1.6510	0.473E-8	4.1889	0.471E-8	4.1923

Conclusions from tables 2 and 3: 1. for fine spatial grids, the convention method ("Compact A") got lower accuracy; 2. "Compact B" and "Compact C" obtain high order accuracy, "Compact C" is slightly better.

6.3 Computations of the Driven Flow in a Square Cavity

We consider the unsteady viscous incompressible fluid flow problem driven by the shearing force in a two dimensional unit square cavity. The control equations are the unsteady viscous incompressible Navier–Stokes equations (3.1)(3.2). Computation area: $0 < x < 1, 0 < y < 1$. Boundary velocity:

$$\boldsymbol{V}_\Gamma = \begin{cases} (1,0)^T & \text{for } 0 < x < 1, y = 1 \\ (0,0)^T & \text{for } 0 < x < 1, y = 0 \text{ and } x = 0, 0 < y < 1 \\ & \quad\quad\text{and } x = 1, 0 < y < 1 \end{cases} \tag{6.6}$$

$\Delta x = 1/N, \Delta y = 1/M, M = N$, Re=$1/\nu$. ∇_H^2 in (4.3) adopts the five–point central difference.

In all calculations, we use the iterative pressure Poisson equation method (5.2)–(5.4) for the pressure. To solve (5.3), we use the multigrid method one loop (grid from fine to coarse, then from coarse to fine), thus the algorithm likes an integrated multigrid procedure. (for such a 'multigrid method', ∇_h^2 is the 'fine grid', ∇_H^2 is the 'coarse grid' (in the 'first–grid')), (for the multigrid method, if we change words 'fine grid' to 'more accurate method', words 'coarse grid' to 'less accurate method', then it might be called a generalized multigrid method); The steady solutions are obtained when t large enough. We denote

Scheme CD-V: the staggered mesh compact difference–finite volume scheme (2.3)–(2.5) (3.5)–(3.7) (3.9)–(3.15) (3.16)

Scheme CD: the staggered mesh compact difference scheme (3.5)–(3.16)

We compared the numerical results of CD-V with those of Ghia's in [2] for Re=100 ($\nu = 0.01$). The accuracy of Ghia's with $N = 128$ is better than the new scheme CD-V with $N = 8$, but much worse than CD-V with $N = 16$. This shows the new scheme's fourth order accuracy.

For Re=1000, $N = 256$, scheme CD-V, besides $u_{\min}$, some selected $(j, u_{N/2,j-\frac{1}{2}})$ are (14,−0.17589); (28,−0.31345); (43,−0.38799); (44,−0.38852); (45,−0.38849); (80,−0.24776); (112,−0.12531); (144,−0.00142); (176,0.13136); (208,0.28591); (245,0.48211); (246,0.50441); (247,0.53114); (252,0.73939), some selected $(i, v_{i-\frac{1}{2},N/2})$ are (14,0.25758); (27,0.34498); (40,0.37681); (41,0.37695); (42,0.37675);

(64,0.30944); (96,0.16270); (128,0.02790); (160,−0.10862); (192,−0.25114); (224,−0.46259); (232,−0.525314); (233,−0.526999); (234,−0.526446); (240,−0.46593); (248,−0.24472).

Extrema in the last eight lines of tables 4 and 5 are obtained from interpolation of Re=1000 solutions of schemes CD-V and CD.

TABLE 4 Extrema of velocity profiles along centerlines (Re=1000)

	$u_{\min}^{(x=0.5)}$	$y_{\min}$	$v_{\max}^{(y=0.5)}$	$x_{\max}$	$v_{\min}^{(y=0.5)}$	$x_{\min}$
Ghia [2]	−0.3829	0.171875	0.37095	0.15625	−0.51550	0.90625
Zhang [7]	−0.39009	0.16992	0.37847	0.15820	−0.52839	0.90820
Bruneau [1]	−0.3764	0.1602	0.3665	0.1523	−0.5208	0.9102
CD-V,N=256	−0.3885729	0.1716965	0.3769494	0.1578361	−0.5270795	0.9092451
CD-V,N=128	−0.3885091	0.1717298	0.3768988	0.1578476	−0.5269636	0.9092524
CD-V,N=64	−0.3874597	0.1722462	0.3759533	0.1580876	−0.5250540	0.9090746
CD-V,N=32	−0.3818458	0.1773832	0.3709188	0.1603107	−0.5114055	0.9051549
CD, N=256	−0.3840607	0.1724296	0.3721873	0.1586994	−0.5213952	0.9089328
CD, N=128	−0.3795204	0.1732002	0.3674140	0.1595922	−0.5156458	0.9086300
CD, N=64	−0.3705614	0.1751717	0.3585948	0.1614637	−0.5043853	0.9078775
CD, N=32	−0.3519381	0.1822162	0.3401250	0.1667745	−0.4761842	0.9032001

* Extrema of Ghia[2], Zhang[7] and Bruneau[1] in tables 4 and 5 are obtained on grid points, absolute values of these extrema should be a little smaller

TABLE 5 Extrema of stream function (Re=1000)

	Primary vortex	Secondary vortex (BL)	Secondary vortex (BR)
	$\psi_{\min}$ (location x, y)	$\psi_{\max}$ (location x, y)	$\psi_{\max}$ (location x, y)
Ghia [2]	−.117929 (.5313,.5625)	.000231129 (.0859,.0781)	.00175102 (.8594,.1094)
Zhang [7]	−.1193 (.5313,.5664)	.000235 (.0820,.0781)	.00174 (.8633,.1133)
Bruneau[1]	−.1163 (.5313,.5586)	.000325 (.0859,.0820)	.00191 (.8711,.1094)
CD-V,256	−.118938(.530789,.56524)	.0002335(.08327,.078095)	.0017297(.86404,.11181)
CD-V,128	−.118925(.530785,.56526)	.0002333(.08325,.078105)	.0017294(.86345,.11149)
CD-V, 64	.118691(.530796,.56552)	.0002309(.08319,.077927)	.0017238(.86414,.11208)
CD-V, 32	−.117567(.530532,.56810)	.0002108(.08318,.076787)	.0017243(.86208,.11363)
CD,N=256	−.117741(.531154,.56559)	.0002234(.08298,.077663)	.0016898(.86449,.11183)
CD,N=128	−.116541(.531521,.56596)	.0002135(.08265,.077228)	.0016501(.86495,.11185)
CD, N=64	−.114019(.532291,.56689)	.0001930(.08182,.076336)	.0015680(.86600,.11210)
CD, N=32	−.109233(.533577,.57032)	.0001483(.08038,.073383)	.0014351(.86615,.11340)

At two upper corner points (0,1) and (1,1), u discontinues, the finite volume scheme decreases the errors of the compact scheme. To show this, we use the result of the scheme CD with $N = 256$ as an "accurate" solution, interpolate it to a coarse grid $N_1 \times N_1$, use four grid point (perpendicular for u, horizontal for v) polynomial interpolation, get $\boldsymbol{V}_{(N_1)}$. For Re=1000, at left upper corner, $i = 1, j = N_1$: $\mathrm{div}_h^{\text{CD-V}}\boldsymbol{V}_{(N_1)}$, $\mathrm{div}_{h_1}^{\text{CD}}\boldsymbol{V}_{(N_1)}$ are $-1.309, -12.42$ for $N_1 = 64$; 2.352, -17.61 for $N_1 = 128$; at right upper corner, $i = N_1, j = N_1$, they are 1.729, 12.99 for $N_1 = 64$; -1.619, 18.66 for $N_1 = 128$. For Re=7500, left-upper: $i = 1, j = N_1$, they are 0.03094, -10.21 for N_1=64; -1.974, -23.84 for $N_1 = 128$; right-upper: $i = N_1, j = N_1$, they are -3.491, 4.877 for $N_1 = 64$; 7.565, 31.83 for $N_1 = 128$. $\mathrm{div}_h^{\text{CD-V}}\boldsymbol{V}_{(N)}$ is defined by (2.3)–(2.5), $\mathrm{div}_h^{\text{CD}}\boldsymbol{V}_{(N)}$ is defined by (3.8)(3.15), $h_1 = 1/N_1$.

For the square–driven cavity with the boundary condition (6.6), Bruneau[1] "conjecture that beyond Re=R_c with 5000 <R_c < 7500 there is not a steady laminar solution any more and the transition to turbulence occurs when small eddies develop along the walls." (pp. 408–412 of [1]). Liu[11][4] gets the same conclusion. While we use the same boundary condition (6.6) and Re=7500, $N = 256$, schemes CD-V and CD, after a period calculation, the numerical solutions approximate steady. Zhang[7] and Ghia[2] also obtained steady solution.

To decide whether the solution should be steady or not, we start the calculation with $N = 512$, alter $(3.16)_2$ to (3.18). The solution is still not convergent so far. Its amplitude of the velocity at the geometric center is now only about 0.04 percent of that in [11]. We will continue to study on this, with larger N, different boundary formulations, nonuniform grid[15][14][13]. We are also performing the calculation for Re=10000, $N = 512$.

7. CONLUSIONS

A new scheme for the incompressible Navier–Stokes equations is proposed. It uses fourth order accurate staggered mesh compact differences or third order accurate staggered mesh upwind compact differences for the momentum equations, and a fourth order accurate integral type finite volume scheme for the continuity equation (truncation errors are given in §3.4). A new set of intermediate boundary conditions of the Runge–Kutta method is described. We use these boundary conditions ("C"), those given by [10] ("B") and the convention method ("A") for a linear scalar equation. All calculations for this case obtain high order accurate solutions, "A" is not bad, "B" is a little better than "C". For a 2D traveling wave calculation, "B" and "C" obtain higher than fourth order accuracy. While "A" gets lower order accurate. "C" is slightly better than "B".

Truncation errors of the discretizations are given in §3.4.

The compact difference–finite volume scheme (CD-V) proposed here achieves high order accuracy for the incompressible fluid flows, its numerical results are better than those obtained by the staggered mesh scheme (CD) (see tables 4 and 5, for Re=1000), and very much better than those of Ghia's in [2], and much better than the results of Bruneau's in [1].

At the location where the boundary velocity discontinues, the finite volume scheme decreases the error of the compact scheme,see the paragraph under table 5.

For Re=7500, we get steady solutions. This supports the conclusion that the Hopf bifurcation point R_c is not lower than 7500. For Re=7500, when there is a perturbation for a steady result, our numerical solution approximates the convergent pattern with periodical oscillations with the time, the amplitude is gradually vanish, the period is about 6.6.

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